

The Automorphism Group of a Domain with an Exponentially Flat Boundary Point¹²

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Abstract: We study the automorphisms group action on a bounded domain in \mathbb{C}^n having a boundary point that is exponentially flat. Such a domain typically has a compact automorphism group. Our results enable us to generate many new examples.

0 Introduction

A domain Ω in \mathbb{C}^n is a connected, open set. An *automorphism* of Ω is a bi-holomorphic self-map. The collection of automorphisms forms a group under the binary operation of composition of mappings. The topology on this group is uniform convergence on compact sets, or the compact-open topology. We denote the automorphism group by $\text{Aut}(\Omega)$.

Although domains with *transitive automorphism group* are of some interest, they are relatively rare (see [HEL]). A geometrically more natural condition to consider, and one that gives rise to a more robust and broader class of domains, is that of having *non-compact automorphism group*. Clearly a domain has non-compact automorphism group if there are automorphisms $\{\varphi_j\}$ which have no subsequence that converges to an automorphism. The following proposition of Henri Cartan is of particular utility in the study of these domains:

Proposition 0.1 *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Then Ω has non-compact automorphism group if and only if there are a point $X \in \Omega$, a point $P \in \partial\Omega$, and automorphisms φ_j of Ω such that $\varphi_j(X) \rightarrow P$ as $j \rightarrow \infty$.*

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We refer the reader to [NAR] for discussion and proof of Cartan's result.

A point P in $\partial\Omega$ is called a *boundary orbit accumulation point* if there is an $X \in \Omega$ and automorphisms φ_j of Ω such that $\lim_{j \rightarrow \infty} \varphi_j(X) = P$.

In the paper [GRK1], we considered the domain

$$\Omega_\infty = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + 2 \exp(-1/|z_2|^2) < 1\}. \quad (*)$$

We showed that this domain has *compact automorphism group*. In particular, the only automorphisms (i.e., biholomorphic self-maps) of Ω are the rotations in each variable separately. We note that, unlike the domains

$$\Omega_m \equiv \{(z_1, z_2); |z_1|^2 + |z_2|^{2m} < 1\}, m = 1, 2, \dots,$$

which have all points of the form $(e^{i\theta}, 0)$ as boundary orbit accumulation points, Ω_∞ has *no* boundary orbit accumulation points.

An example such as this bears directly on the Greene-Krantz conjecture:

Conjecture: Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain. If P is a boundary orbit accumulation point, then P must be a point of finite type in the sense of Kohn/Catlin/D'Angelo (see [KRA1] for this concept).

This conjecture is not known to be true in its full generality. But evidence for its correctness is provided, for instance, in [KIM] and [KIK].

The domain Ω_∞ in (*) can be analyzed like this: If there is an $X \in \Omega_\infty$ and a $P = (p_1, p_2) \in \partial\Omega_\infty$ with $p_2 \neq 0$ so that $\lim_{j \rightarrow \infty} \varphi_j(X) = P$ for some automorphisms φ_j , then P is strongly pseudoconvex. The theorem of Bun Wong and Rosay (see [WON], [ROS]) then tells us that Ω_∞ must be biholomorphic to the ball. But a result of Bell/Boas [BEB] tells us that such a biholomorphism must extend smoothly to the boundary. That is impossible, since Ω_∞ has a circle $\{(e^{i\theta}, 0)\}$ of weakly pseudoconvex points in the boundary while the ball B is strongly pseudoconvex. If instead $p_2 = 0$, then a delicate analysis—see the proof of Proposition 1.3 below—shows that it is impossible for an orbit $\{\varphi_j(X)\}$ to accumulate at P . Thus we conclude, using Proposition 0.1, that $\text{Aut}(\Omega_\infty)$ must be compact.

Of course the boundary points $(e^{i\theta}, 0)$ of the domain Ω_∞ in (*) are of infinite type. It is important to have examples like this at hand to aid in the study of the Greene-Krantz conjecture.

The purpose of this paper is to provide a good many further examples of such domains. We provide domains with different types of geometry in different dimensions. All have compact automorphisms group.

In general it is quite difficult to produce examples of any kind in the study of automorphism groups of domains in \mathbb{C}^n . The techniques presented in this paper may prove useful in other contexts.

1 Some Principal Results

The following simple technical result will be useful. I thank John P. D'Angelo for helpful discussions of the idea.

Lemma 1.1 *Let F be a holomorphic function on the unit ball B in \mathbb{C}^n , $n \geq 2$, which extends continuously to \overline{B} . Assume that $|F|$ on ∂B equals a positive constant c . Then F is identically constant.*

Proof: Let P be a boundary point of the ball. Let \mathbf{d} be an analytic disc whose image lies in a complex line which intersects ∂B transversally. Also suppose that the boundary circle of this analytic disc lies in ∂B —in other words, the analytic disc is just a complex line intersected with \overline{B} . Assume that \mathbf{d} is very close to P . Then $|F|$ restricted to \mathbf{d} will assume values that are uniformly very close to c . But $|F|$ restricted to \mathbf{d} is a function of one complex variable that satisfies the hypotheses of the lemma. So F restricted to \mathbf{d} must be a finite Blaschke product with values which in modulus are very close to c . It follows that F can have no zeros, so F restricted to \mathbf{d} must be a constant. Since this statement must hold on all such analytic discs closed to P , we may conclude that F is constant. \square

Remark 1.2 In point of fact, this lemma is true on any smoothly bounded domain. For such a domain will always have a relatively open boundary neighborhood that is strongly convex. And the argument just presented will be valid on that neighborhood.

Proposition 1.3 *Consider the domain*

$$\tilde{\Omega} = \{z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n : |z_1|^2 + |z_2|^2 + \dots + |z_{n-1}|^2 + \psi(|z_n|) < 1\},$$

where ψ is a real-valued, even, smooth, monotone-and-convex-on- $[0, \infty)$ function of a real variable with $\psi(0) = 0$ that vanishes to infinite order at 0. Then there do not exist a point $X \in \tilde{\Omega}$ and automorphisms φ_j of $\tilde{\Omega}$ so that $\lim_{j \rightarrow \infty} \varphi_j(X) = (1, 0, \dots, 0) \in \partial \tilde{\Omega}$.

Remark 1.4 Observe that points ζ of $\partial \tilde{\Omega}$ with $\zeta_n \neq 0$ are strongly pseudoconvex (as a calculation shows). Such a point cannot be an orbit accumulation point because then, by the Bun Wong/Rosay theorem, the domain would have to be biholomorphic to the unit ball.

By contrast, the boundary points of the form $(e^{i\theta}, 0, \dots, 0)$, $(0, e^{i\theta}, 0, \dots, 0)$, \dots , $(0, 0, \dots, e^{i\theta}, 0)$ are of infinite type. We state a result for the particular boundary point $(1, 0, 0, \dots, 0)$, but in fact the analysis applies to any boundary point of the form $(e^{i\theta}, 0, 0, \dots, 0)$, $(0, e^{i\theta}, 0, \dots, 0)$, \dots , $(0, 0, \dots, e^{i\theta}, 0)$, $0 \leq \theta \leq 2\pi$.

Of course an example of a real function ψ as in the statement of the theorem is $\lambda(t) = 2 \exp(-1/t^2)$.

Proof of Proposition 1.3: Seeking a contradiction, let us suppose that there is a point $X \in \tilde{\Omega}$ and a sequence of automorphisms φ_j such that $\varphi_j(X) \rightarrow (1, 0, \dots, 0) \in \partial\tilde{\Omega}$.

The set of weakly pseudoconvex points in the boundary is

$$S = \{z \in \partial\tilde{\Omega} : z_n = 0\}.$$

All other boundary points are strongly pseudoconvex. Since the work of Bell/Boas [BEB] tells us that automorphisms extend smoothly to the boundary, we may conclude that any automorphism φ will preserve S . Thus it will also preserve the ball

$$\mathbf{e} = \{(z_1, z_2, \dots, z_{n-1}, 0) : \sum_{j=1}^{n-1} |z_j|^2 \leq 1\}.$$

We see then that the restriction of φ to \mathbf{e} will be an automorphism of the unit ball in \mathbb{C}^{n-1} . After composing each φ_j with rotations in the z_1, z_2, \dots, z_{n-1} variables, we may take it that

$$\begin{aligned} \varphi_j(z_1, z_2, z_3, \dots, z_{n-1}, 0) &= \\ &\left(\frac{z_1 - a_j^1}{1 - \bar{a}_j^1 z_1}, \frac{\sqrt{1 - |a_j^2|^2} z_2}{1 - \bar{a}_j^2 z_1}, \frac{\sqrt{1 - |a_j^3|^2} z_3}{1 - \bar{a}_j^3 z_1}, \dots, \frac{\sqrt{1 - |a_j^{n-1}|^2} z_{n-1}}{1 - \bar{a}_j^{n-1} z_1}, 0 \right). \end{aligned}$$

for some $a_j^\ell \in \mathbb{C}$, $|a_j^\ell| < 1$, $\ell = 1, \dots, n-1$. By imitating the proof of the classical result (see [KRA1]) about automorphisms of circular domains, we may see that each φ_j commutes with rotations in the z_n variable. It follows that each φ_j preserves every disc of the form

$$\mathbf{f}_\alpha \equiv \{(\alpha, \zeta) : |\alpha|^2 + \psi(|\zeta|) < 1\},$$

for some fixed $\alpha \in \mathbb{C}^{n-1}$ with $|\alpha| < 1$. Of course here $\zeta \in \mathbb{C}$. By rotational invariance, the image of \mathbf{f}_α under φ_j must be a disc of the form

$$\varphi_j(\mathbf{f}_\alpha) = \{(\varphi_j(\alpha, 0), \zeta) : |\varphi_j(\alpha, 0)|^2 + \psi(|\zeta|) < 1\}.$$

Furthermore, φ_j sends the center of \mathbf{f}_α to the center of the image disc. We conclude then that φ_j must have the form

$$\begin{aligned} (z_1, z_2, z_3, \dots, z_n) &\longmapsto \\ &\left(\frac{z_1 - a_j^1}{1 - \bar{a}_j^1 z_1}, \frac{\sqrt{1 - |a_j^2|^2} z_2}{1 - \bar{a}_j^2 z_1}, \frac{\sqrt{1 - |a_j^3|^2} z_3}{1 - \bar{a}_j^3 z_1}, \dots, \frac{\sqrt{1 - |a_j^{n-1}|^2} z_{n-1}}{1 - \bar{a}_j^{n-1} z_1}, z_n \cdot \lambda_j(z_1, \dots, z_{n-1}) \right), \end{aligned}$$

for λ_j holomorphic.

Thus we see that

$$\begin{aligned} & \left| \frac{z_1 - a_j^1}{1 - \bar{a}_j^1 z_1} \right|^2 + \left| \frac{\sqrt{1 - |a_j^2|^2} z_2}{1 - \bar{a}_j^2 z_1} \right|^2 + \left| \frac{\sqrt{1 - |a_j^3|^2} z_3}{1 - \bar{a}_j^3 z_1} \right|^2 + \cdots + \\ & \left| \frac{\sqrt{1 - |a_j^{n-1}|^2} z_{n-1}}{1 - \bar{a}_j^{n-1} z_1} \right|^2 + \psi(z_n \cdot \lambda_j(z_1, \dots, z_{n-1})) < 1. \quad (\dagger) \end{aligned}$$

This may be rewritten as

$$\begin{aligned} |z_n| < \frac{1}{|\lambda_j(z_1, z_2, \dots, z_{n-1})|} \cdot \psi^{-1} \left(1 - \left| \frac{z_1 - a_j^1}{1 - \bar{a}_j^1 z_1} \right|^2 \right. \\ \left. - \left| \frac{\sqrt{1 - |a_j^2|^2} z_2}{1 - \bar{a}_j^2 z_1} \right|^2 - \left| \frac{\sqrt{1 - |a_j^3|^2} z_3}{1 - \bar{a}_j^3 z_1} \right|^2 - \cdots - \left| \frac{\sqrt{1 - |a_j^{n-1}|^2} z_{n-1}}{1 - \bar{a}_j^{n-1} z_1} \right|^2 \right) \end{aligned}$$

But we also know that

$$|z_n| < \psi^{-1} (1 - |z_1|^2 - |z_2|^2 - \cdots - |z_{n-1}|^2) .$$

The only possible conclusion is that

$$\begin{aligned} & \frac{\psi^{-1} \left(1 - \left| \frac{z_1 - a_j^1}{1 - \bar{a}_j^1 z_1} \right|^2 - \left| \frac{\sqrt{1 - |a_j^2|^2} z_2}{1 - \bar{a}_j^2 z_1} \right|^2 - \left| \frac{\sqrt{1 - |a_j^3|^2} z_3}{1 - \bar{a}_j^3 z_1} \right|^2 - \cdots - \left| \frac{\sqrt{1 - |a_j^{n-1}|^2} z_{n-1}}{1 - \bar{a}_j^{n-1} z_1} \right|^2 \right)}{\psi^{-1} (1 - |z_1|^2 - |z_2|^2 - \cdots - |z_{n-1}|^2)} \\ & = |\lambda_j(z_1, z_2, \dots, z_{n-1})|. \quad (\star) \end{aligned}$$

Now we let $|(z_1, z_2, \dots, z_{n-1})| \rightarrow 1$. Since ψ^{-1} vanishes to infinite order, but the two arguments of ψ^{-1} vanish to finite order, we can only conclude that the lefthand side of (\star) tends to some positive constant γ . Hence

$$|\lambda_j(z_1, z_2, \dots, z_{n-1})| = \gamma$$

on the boundary of the unit ball in $(n-1)$ -dimensional complex space. By our lemma, we conclude that λ is a constant function. It follows that the mappings φ_j cannot exist (as automorphisms of $\tilde{\Omega}$). Hence our proof is complete. \square

Remark 1.5 In fact the proposition shows that there are uncountably many biholomorphically distinct domains that satisfy the conclusion of the assertion. For one has great freedom in selecting the function ψ , and the theory of the Chern-Moser invariants establishes the biholomorphic distinctness for the different choices.

Next we have:

Proposition 1.6 *Consider the domain*

$$\tilde{\tilde{\Omega}} = \{z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n : |z_1|^{2m_1} + |z_2|^{2m_2} + \dots + |z_{n-1}|^{2m_{n-1}} + \psi(|z_n|) < 1\},$$

where the m_j are positive integers and where ψ is a real-valued, even, smooth, monotone-and-convex-on-[0, ∞) function of a real variable with $\psi(0) = 0$ that vanishes to infinite order at 0. Then $\tilde{\tilde{\Omega}}$ has compact automorphism group.

Proof: If all the the m_j are greater than 1 then it is easy to calculate that all of the circles

$$\{(e^{i\theta}, 0, \dots, 0) : 0 \leq \theta \leq 2\pi\}, \{(0, e^{i\theta}, 0, \dots, 0) : 0 \leq \theta \leq 2\pi\}, \dots, \\ \{(0, 0, \dots, 0, e^{i\theta}) : 0 \leq \theta \leq 2\pi\}$$

consist of weakly pseudoconvex points. In fact points on any of those circles have $(n-1)$ weakly pseudoconvex directions. The boundary points with all coordinates nonvanishing are strongly pseudoconvex. Thus points of the latter type form a relatively open and dense set in the boundary. One may argue that, if the automorphism group were noncompact and there were a point $X \in \tilde{\tilde{\Omega}}$ and a point $P \in \partial\tilde{\tilde{\Omega}}$ with $\varphi_j(X) \rightarrow P$ for some automorphisms φ_j , then at least one of these circles would be moved towards P by the φ_j . As a result, weakly pseudoconvex boundary points would be mapped to strongly pseudoconvex boundary points. And that is impossible.

So the only case of interest is that where at least one of the m_j is equal to 1. Say for specificity that $m_1 = 1$. In that case points of the form $(e^{i\theta}, 0, \dots, 0)$ have $(n-1)$ weakly pseudoconvex directions, whereas points on the other indicated circles have one weakly pseudoconvex direction. As before, points with all coordinates nonvanishing are strongly pseudoconvex. So we can argue as above to see that the automorphism group must be compact. \square

Another type of domain that we may consider is described in the next proposition:

Proposition 1.7 *Consider a domain of the form*

$$\Omega' = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : |z_1|^2 + 2 \exp(-1/(|z_2|^2 + \dots + |z_n|^2))\} < 1.$$

Then Ω' must have compact automorphism group.

Proof: As usual, points with $z_2 \neq 0, z_3 \neq 0, \dots, z_n \neq 0$ are strongly pseudoconvex. By the Bun Wong/Rosay theorem, they cannot be orbit accumulation points.

If a point of the form $(e^{i\theta}, 0, \dots, 0)$ is an orbit accumulation point, then we may argue as in the proof of the first proposition. The main difference is that line (\dagger) is replaced by

$$\left| \frac{z_1 - a_j^1}{1 - \overline{a_j^1} z_1} \right|^2 + \rho(z_2, z_3, \dots, z_n) \cdot \lambda_j(z_1) < 1,$$

where ρ is a unitary rotation in $(n - 1)$ variables. The argument is now completed as before. \square

2 Concluding Remarks

There is no Riemann mapping theorem in several complex variables. Thus we seek other means of contrasting and comparing domains in \mathbb{C}^n . The automorphism group and its associated features have proved to be powerful and flexible invariants in this study. Certainly the Levi geometry of boundary orbit accumulation points has been studied extensively ([GRK2] and [ISK]). The results of this paper fit into that program.

We hope to continue these studies in future papers.

REFERENCES

- [BEB] S. Bell and H. Boas, Regularity of the Bergman projection in weakly pseudoconvex domains, *Math. Ann.* 257(1981), 23–30.
- [CHM] S.-S. Chern and J. Moser, Real hypersurfaces in complex manifolds, *Acta Math.* 133(1974), 219–271.
- [GRK1] R. E. Greene and S. G. Krantz, Techniques for studying automorphisms of weakly pseudoconvex domains, *Several Complex Variables* (Stockholm, 1987/1988), Math. Notes 38, Princeton University Press, Princeton, NJ, 1993, 389–410.
- [GRK2] R. E. Greene and S. G. Krantz, Biholomorphic self-maps of domains, *Complex Analysis II* (C. Berenstein, ed.), Springer Lecture Notes, vol. 1276, 1987, 136–207.
- [HEL] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.
- [ISK] A. Isaev and S. G. Krantz, Domains with non-compact automorphism group: a survey, *Adv. Math.* 146(1999), 1–38.
- [KIM] K.-T. Kim, Domains in \mathbb{C}^n with a piecewise Levi flat boundary which possess a noncompact automorphism group, *Math. Ann.* 292(1992), 575–586.
- [KIK] K.-T. Kim and S. G. Krantz, Complex scaling and domains with non-compact automorphism group, *Illinois Journal of Math.* 45(2001), 1273–1299.
- [KRA1] S. G. Krantz, *Function Theory of Several Complex Variables*, 2nd ed., American Mathematical Society, Providence, RI, 2001.
- [NAR] R. Narasimhan, *Several Complex Variables*, University of Chicago Press, Chicago, 1971.
- [ROS] J.-P. Rosay, Sur une caracterization de la boule parmi les domaines de \mathbb{C}^n par son groupe d’automorphismes, *Ann. Inst. Four. Grenoble* XXIX(1979), 91–97.
- [WON] B. Wong, Characterizations of the ball in \mathbb{C}^n by its automorphism group, *Invent. Math.* 41(1977), 253–257.

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